**Indian Statistical Institute, Bangalore Centre**
**B.Math. (III Year) : 2015-2016**
**Semester II : Semestral Examination**
**Probability III (Stochastic Processes)**

29.04.2016 Time: 3 hours. Maximum Marks : 100

*Note:* Notation and terminology are understood to be as used in class. State clearly the results you are using in your answers.

1. ( 15 marks ) Box I and Box II together contain a total of $N$ balls. At each trial, a ball is chosen at random from among the $N$ balls; also, independently, one of the boxes is selected such that the probability of selecting Box I is $p$, and that of selecting Box II is $1-p$, where $0 < p < 1$; then the ball chosen is placed in the box selected. Let $X_n$ denote the number of balls in Box I at the end of the $n-$th trial. Find the transition probability matrix of $\{X_n\}$.

2. ( 15 marks ) $\{X_n : n = 0, 1, 2, \cdots\}$ is a time-homogeneous Markov chain on $S = \{1, 2, 3, 4\}$ with transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

Find the stationary probability distribution for $\{X_n\}$. Is it the limiting probability distribution of $\{X_n\}$ ?

3. ( 15 marks ) Find an irreducible Markov chain $\{X_n\}$ on $S = \{1, 2, 3\}$ such that (i) $d_x < \min\{n \geq 1 : P_{xx}^{(n)} > 0\}$ for all $x \in S$, and (ii) $\{X_n\}$ is not time reversible. Here $d_x$ denotes the period of $x$. (*Hint:* You may try $P_{ij} > 0, P_{ji} = 0$ for suitable $i, j$.)

4. ( 15 marks ) $\{N(t) : t \geq 0\}$ is a time-homogeneous Poisson process with rate $\lambda > 0$. Let $k \geq 2$ be an integer. Find the probability that the time between $k-$th and $(k-1)-$th arrivals is at least twice as much as that between $(k-1)-$th and $(k-2)-$th arrivals.

5. ( 20 marks ) Claims to an insurance company arrive at times $\{W_i\}$ according to a time-homogeneous Poisson process $N(\cdot)$ with rate $\lambda > 0$.

For $i \geq 1$, claim arriving at $W_i$ involves an instantaneous payment of $X_i > 0$ by the company. $\{W_i\}$ and $\{X_j\}$ are assumed to be independent families of random variables. $S(t) = \sum_{i=1}^{N(t)} e^{-rW_i} X_i$ denotes the discounted value (at time 0) of the cumulative claim amount over the period $[0, t]$, where $r > 0$ is the discount rate. Find $E(S(t)), t \geq 0$.

6. ( 20 marks ) Let $\{P_{ij}(t) : t \geq 0,\ i, j \in S\}$ denote the transition probability function of a birth and death process on $S := \{0, 1, 2, \cdots\}$, with infinitesimal birth rates $\lambda_i > 0,\ i \geq 0$, and infinitesimal death rates $\mu_0 = 0, \mu_i > 0,\ i \geq 1$. Assume that for $j = 0, 1, 2, \cdots$

$$\sup_{k \neq j-1, j, j+1} \frac{1}{h} P_{kj}(h) \to 0, \text{ as } h \downarrow 0.$$

Show that $\{P_{ij}(t)\}$ satisfies the following system of ODE's: For fixed $i \geq 0$,

$$\begin{aligned} P'_{i0}(t) &= -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t),\ t > 0, \\ P'_{ij}(t) &= \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j + \mu_j) P_{ij}(t) + \mu_{j+1} P_{i,j+1}(t),\ t > 0, j \geq 1, \end{aligned}$$

with the initial conditions $P_{ij}(0) = \delta_{ij}$. (*Hint:* Split the time interval $[0, t+h]$ into $[0, t]$ and $[t, t+h]$.)